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A Note on the Artuso–Aurell–Cvitanovic Approach to the Feigenbaum Tangent Operator

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In this note we explain the rigorous mathematical arguments underlying some recent work of Artuso, Aurell and Cvitanovic on the Feigenbaum tangent operator. In particular, we attempt to clarify the advantages of introducing zeta functions through the ideas of Ruelle and Grothendieck.

KEY WORDS: Feigenbaum conjectures; zeta functions; transfer operators; hyperbolic fixed points.

INTRODUCTION

The purpose of this note is to clarify the rigorous mathematical arguments underlying some recent work of Artuso *et al.*[1] These authors were concerned with those parts of the well-known Feigenbaum conjectures for the period-doubling phenomena of unimodular maps which dealt with the spectrum of the derivative of the Feigenbaum operator.

It had been observed by various authors that this linear operator took the form of a very special kind of Ruelle-type transfer operator for a "cookie cutter."

These special features were used by Artuso *et al.* to develop an approach (based on zeta functions and periodic points for the "cookie cutter") to understanding the spectrum.

In this note we shall attempt to clarify these ideas and, in particular, demonstrate that this yields an alternative way to show the existence of a maximal positive eigenvalue and hyperbolicity of the operator, and an efficient and rigorous method for calculating the value of the maximal eigenvalue (given some information about the general form of the Feigen-

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baum "fixed point" function in the form of weights for a finite number of periodic points) of the associated "cookie cutter."

We emphasize that we are *not* attempting to make a new conceptual contribution to the theory, merely reformulating the ideas of Artuso *et al.* in such a way as to demonstrate that their approach is intrinsically mathematically rigorous.

1. COOKIE CUTTERS AND THEIR SPECTRA

Consider the following construction: Let $I = [a, b]$ denote a closed interval in the real line and let $I_1 = [a, a']$ and $I_2 = [b', b]$ be two subintervals with $a < a' < b' < b$.

Let $F_i: I_i \to I$, $i = 1, 2$, be two real analytic surjective expanding maps (with $\|F_i'\|_\infty > 1$) and we write $F: I_1 \cup I_2 \to I$.

The maps F_i have inverses $f_i: I \to I_i$, $i = 1, 2$, such that (Fig. 1):

(i) f_1, f_2 are real analytic.

(ii) f_1, f_2 are contractions (with $0 < |f_i'(z)| < 1$, $i = 1, 2$).

(iii) f_1, f_2 are surjective.

Since the maps f_i $(i = 1, 2)$ are *real* analytic, they have *complex* analytic extensions to some open neighborhood $I \subseteq U \subseteq \mathbb{C}$ (and we keep the same notation f_i for their extensions to U). Furthermore, by choosing U sufficiently small and using (ii), we see that:

(iv) $\mathrm{cl}(f_i U) \subseteq U$, $i = 1, 2$.

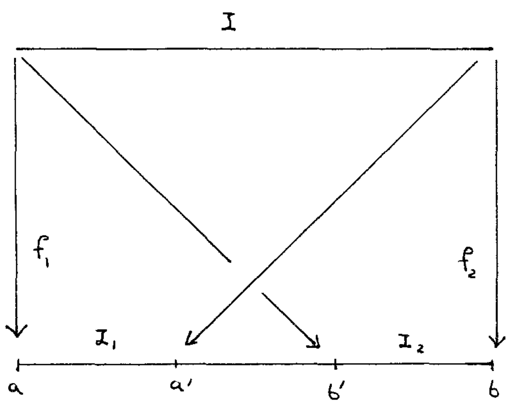

Fig. 1. A cookie cutter.

Let B denote the space of (complex) analytic functions $h: U \to \mathbb{C}$ which have a continuous extension to $\mathrm{cl}(U)$. This is a Banach space with respect to the supremum norm

$$\|g\|_\infty = \sup\{|g(z)| \mid z \in U\}$$

Assume that $w_i: f_i U \to \mathbb{C}$, $i = 1, 2$, are analytic functions which have a continuous extension to $\mathrm{cl}(f_i U)$. We shall also write $w: f_1 U \cup f_2 U \to \mathbb{C}$, where $w \mid f_i U = w_i$, $i = 1, 2$.

Definition. The *transfer* (or *Ruelle*) *operator* $L: B \to B$ is a linear operator defined by

$$(Lh)(z) = w_1(f_1 z) \cdot h(f_1 z) + w_2(f_2 z) \cdot h(f_2 z)$$

The operator is clearly bounded and compact (seen with the aid of Montel's theorem). However, it is easy to show the *stronger* result that the operator takes the following explicit form:

Lemma 1. We can write

$$Lh(z) = \sum_{n=0}^{+\infty} v_n(h) \cdot (f_1 z - a)^n + \sum_{n=0}^{+\infty} u_n(h) \cdot (f_2 z - b)^n$$

where

(i) $z \mapsto (f_1 z - a)^n$, $(f_2 z - b)^n$ are complex functions and

(ii) $$h \mapsto v_n(h) = \frac{1}{2\pi i} \int_\Gamma \frac{(w_1 h)(\xi)}{(\xi - f_1 z)^{n+1}} \, d\xi$$

$$h \mapsto u_n(h) = \frac{1}{2\pi i} \int_\Gamma \frac{(w_2 h)(\xi)}{(\xi - f_2 z)^{n+1}} \, d\xi \quad \text{are linear functionals}$$

and where $\Gamma \subseteq U$ is a closed curve encircling $\mathrm{cl}(f_1 U)$, $\mathrm{cl}(f_2 U)$ (Fig. 2).

[Of course, these expansions are only valid within the radius of convergence of the series. Therefore they should be interpreted in the sense that we can choose appropriate points a and b and curve Γ, depending on the neighborhood of z. The proof of the expansion is very easy. Simply write $(w_i h)(f_i z)$, $i = 1, 2$, in terms of the Cauchy integral around Γ and then expand the integrand as a power series. In particular, L is a "nuclear" or "trace-class" operator in the sense of Grothendieck.[8]]

Let $\{\beta_i\}_{i=1}^{+\infty}$ denote the eigenvalues of the (compact) operator $L: B \to B$, repeated according to multiplicity.

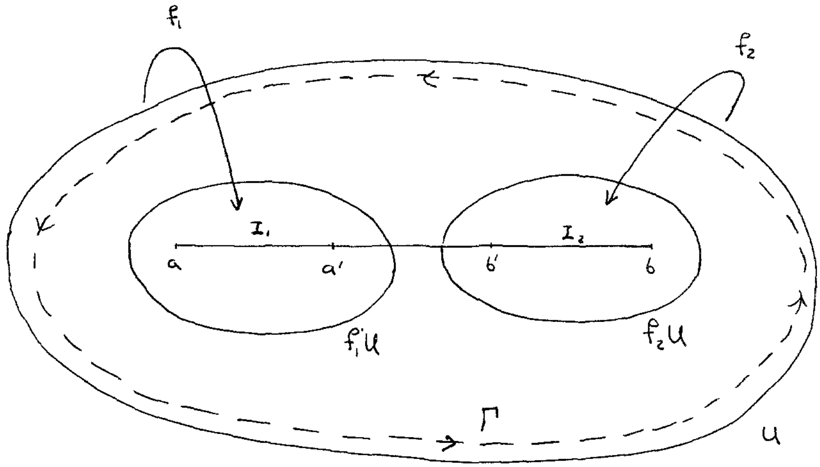

Fig. 2. The analytic domains.

Definition. We define the *determinant* to be the complex function

$$d(z) = \prod_{i=1}^{+\infty} (1 - z\beta_i), \qquad z \in \mathbb{C}$$

The relationship between the spectrum of L and the domain of d is given by the following:

Lemma 2. (i) $d(z)$ converges on a sufficiently small neighborhood $|z| < \varepsilon$, $\varepsilon > 0$.

(ii) $d(z)$ extends as an entire function on $\mathbb{C}$.

(iii) The zeros for $d(z)$ occur at $z = 1/\beta_i$, $i = 1, 2,...$.

(These results are well known from Grothendieck's work.[8])

We make some observations that will be useful later. We may expand $d(z)$ for $|z| < \varepsilon$ by

$$d(z) = 1 + \sum_{n=1}^{+\infty} a_n z^n, \qquad \text{where} \quad a_n = \sum_{i_1 < \cdots < i_n} \beta_{i_1} \cdots \beta_{i_n}$$

The proof of Lemma 2(ii) is actually based on *explicit* bounds on the coefficients a_n in terms of bounds on $\|L_i\|$, where $L = \sum_{i=0}^{+\infty} L_i$ represents the presentation in Lemma 1.

Specifically, if we have $\|L_k\| \leqslant C \cdot \theta^k$, $k \geqslant 0$, then

$$|a_k| \leqslant \frac{C^k \cdot \theta^{k(k-1)/2}}{(1-\theta)(1 \cdot \theta^2) \cdots (1-\theta^k)} = O(C^k \theta^{k^2}) \tag{1.1}$$

(see ref. 7 for details of this estimate).

Here we can choose:

(i) $C = \max\{\|w_1\|_\infty, \|w_2\|_\infty\}$.

(ii) $\theta = \max\{(r_1/R_1)^{1/2}, (r_2/R_2)^{1/2}\}$, where

$$r_1 = \sup\{|f_1 z - a| \mid z \in U\}$$
$$r_2 = \sup\{|f_2 z - b| \mid z \in U\}$$
$$R_1 = \inf\{|\xi - \alpha| \mid \xi \in \Gamma\}$$
$$R_2 = \inf\{|\xi - b| \mid \xi \in \Gamma\}$$

[and in specific cases more intelligent (i.e., smaller) choices of θ are possible].

For an appropriate domain U and curve Γ we expect to make $0 < \theta < 1$. (*N.B.* We can replace a, b by other points in the intervals I_1 and I_2 if need be.)

The most familiar case of transfer operators is when $w_i > 0$. In this context we have the following:

Ruelle Operator Theorem. For $w > 0$ the transfer operators have a maximal (in modulus) simple positive eigenvalue, $\beta_1 > 0$. In particular, if $w = 1/|F'|$, then $\beta_1 \leqslant 1$.[12]

Clearly, if $w < 0$, then the transfer operator has a maximal (in modulus) simple negative eigenvalue.

Definition. In the case where the weight functions w_1, w_2 have values on $I_1 \cup I_2$ with mixed signs we shall say the transfer operator has *sinned.*

Example. Consider linear contrations $f_1(z) = \alpha(z-a)$, $f_2(z) = \beta(z-b)$, where $0 < |\alpha|$, $|\beta| < 1$ and let the weight functions take constant values $w_1(z) = A$, $w_2(z) = B$, where $|A|$, $|B| > 0$.

The domain U can be chosen to be a disc $|z| \leqslant R$, for sufficiently large R, and then we can take $C = \text{Max}\{|A|, |B|\}$ and as $R \to +\infty$, θ^2 can be taken arbitrarily close to $\theta_0^2 = \max\{|\alpha|, |\beta|\}$.

The spectrum in this case can be explicitly computed as $\{A\alpha^n + B\beta^n \mid n \geqslant 0\}$.

The determinant becomes

$$d(z) = \prod_{n=0}^{+\infty} [1 - z^n(A\alpha^n + B\beta^n)]$$

Even in the relative simple case of this example there is discouragingly little evidence of a very general critereon for the existance of a positive maximal eigenvalue for the transfer operator.

If $\alpha = \beta > 0$ and $A > |B| > 0$, then there exists a simple maximal eigenvalue. However, $A > |B| > 0$ is no longer a sufficient condition if we take $\alpha > 0 > \beta$ with $\beta = -\alpha$.

Remark. It is natural to expect that for operators "sufficiently close" to those trivial operators in the example we shall have some information on the spectrum by analytic perturbation theory. One problem here is to quantly "sufficiently close." *The real benefit of introducing determinants $d(z)$ is that we will eventually be able to have very accurate estimates on the coefficients a_n, and so ultimately the eigenvalues β_i.*

Since our principal motivation is the study of the specific case of the Feigenbaum tangent operator (to be described in the next section), this aspect is important.

2. THE FEIGENBAUM TANGENT MAP

The well-known Feigenbaum "conjectures" have three parts:

Part 1. There exists an analytic map $g: [-1, 1] \to \mathbb{R}$ which is a fixed point for the map Φ defined by $(\Phi h)(z) = \alpha h \circ h(z/\alpha)$ [where we assume the normalization $g(0) = 1$, which specifies $\alpha = 1/g(1)$].

Part 2. The tangent map $D_g \Phi$ has a maximal positive eigenvalue $\delta > 0$ (on a suitable Banach space).

Part 3. Aside from the "trivial" eigenvalues α, 1, $1/\alpha$, $1/\alpha^2$,..., the rest of the spectrum of $D_g \Phi$ is strictly within the unit disc.

We refer to ref. 9 for a particularly clear and concise exposition.

All parts have been rigorously proved by Lanford[10] (through a computation-oriented approach). Subsequently, a more conceptual approach to the first part was developed by Epstein,[5] and Campanino *et al.*[3] gave a proof of the second part by introducing an invariant cone (cf. also refs. 4 and 6).

Remarks. (a) Lanford's computational approach yields the numerical values $\alpha = -2.50290...$ and $\delta = 4.6692...$.

(b) The "trivial" eigenvalues α, 1, $1/\alpha$, $1/\alpha^2$,... correspond to eigenfunctions $z \mapsto z^k \cdot g'(z) - [g(z)]^k$, $k \geq 0$.

(c) The even functions [i.e., $h(x)$ such that $h(x) = h(-x)$] are invariant under $D_g\Phi$, and restricting the operator to this space only serves to eliminate spectra from within the set of trivial eigenvalues. For example, α corresponds to the eigenfunction $z \mapsto g'(z) - 1$, which is not an even function [since $z \mapsto g(z)$ is even] and so α does *not* occur as an eigenvalue for $D_g\Phi$ acting on even functions.

(d) The eigenvalue 1 *does* occur for $D_g\Phi$ acting on even functions, has eigenvector $z \mapsto zg'(z) - g(z)$ and eigenprojection $h \mapsto C(h) = (\alpha^2 - 1)\, h(0) - \alpha h(1) \in \mathbb{R}$.

We refer to ref. 9 for more details and references to primary sources.

To calculate the derivative $D_g\Phi$, we need to consider the first-order term in $t \mapsto \Phi[g + t]$, which gives $(D_g\Phi)\, h(z) = \alpha h(g(z/\alpha)) + \alpha g'(g(z/\alpha)) \cdot h(z/\alpha)$.

We shall restrict consideration to the case where h is an even function. We *could* write $h(z) = k(z^2)$ for some k, but since g is even, it is convenient to write $h(z) = k(g^{-1}(z))$. It then follows that we can rewrite the derivative as $(D_g\Phi)h = (Lk) \circ g$, i.e.,

$$h(z) \mapsto \alpha k(z/\alpha) + \alpha g'(g(z/\alpha)) \cdot k(g(z/\alpha)) \qquad \text{for} \quad 1/\alpha \leq z \leq 1$$

(cf. ref. 1 for details of the simple derivation. We learnt of this fact from Sullivan,[14] but we do not know who first introduced it).

Thus, we see that $D_g\Phi$ corresponds to a transfer operator with

$$f_1\colon\ [1/\alpha, 1] \to [1/\alpha, 1/\alpha^2], \qquad f_1(z) = z/\alpha$$

$$f_2\colon\ [1/\alpha, 1] \to [g^{-1}(1/\alpha^2), 1], \qquad f_2(z) = g^{-1}(z/\alpha)$$

and $w_i(z) = (F)'(z)$, $i = 1, 2$.

Since f_1 is orientation reversing and f_2 is orientation preserving (and consequently, $w_2 > 0$, $w_1 < 0$, on the appropriate intervals), we can conclude that the operator L has sinned.

Standing Hypothesis. Unless otherwise stated, we shall henceforth only consider transfer operators with weight functions $w = F'$.

3. ZETA FUNCTIONS

We next try to understand the determinants $d(z)$ in terms of the periodic points of F. This is the key idea in the work of Artuso *et al.*[1]

For each $n \geqslant 1$ there are periodic orbits $\tau = \{x, Fx, ..., F^{n-1}x\}$ with (prime) F-period $|\tau| = n$. We can associate an (unweighted) *zeta function* by the Euler product expression

$$\zeta_0(z) = \prod_\tau (1 - z^{|\tau|})^{-1}$$

By a now standard computation, this complex function has a meromorphic extension $\zeta_0(z) = 1/(1-2z)$.[2]

Definition. For an analytic weight function $w: I_1 \cup I_2 \to \mathbb{R}$ we can define the (weighted) *Ruelle zetà function* by

$$\zeta(z) = \prod_\tau (1 - a_\tau \cdot z^{|\tau|})^{-1}$$

where $a_\tau = \prod_{i=0}^{n-1} w(F^i x)$ is the product of the values of the weight function around the orbit $\{x, Fx, ..., F^{n-1}x\}$.

Lemma 3. (i) $\zeta(z)$ converges to an analytic function on a sufficiently small neighborhood $|z| < \varepsilon$.

(ii) $\zeta(z)$ has a meromorphic extension to $\mathbb{C}$.

(iii) $\zeta(z) = d(z, w/F')/d(z, w)$, where $d(z, \cdot)$ is the determinant for the transfer operator with the appropriate weight function.

These results can be derived from Ruelle's article,[13] although it is perhaps a little easier to understand the ideas from Mayer's later article[11] for the special case of the continued-fraction transformation.

By our "standing hypothesis" we want to take $w = F'$, and so, applying Lemma 3 twice (with choices $w = F'$ and then $w = 1$) gives

$$\zeta(z) = \frac{d(z, 1)}{d(z, F')} \qquad (\text{with } w = F') \tag{3.1}$$

and

$$\zeta_0(z) = \frac{d(z, 1/F')}{d(z, 1)} \qquad (\text{with } w = 1) \tag{3.2}$$

Thus, combining (3.1) and (3.2), we have

$$d(z) = d(z, F') = \prod_\tau (1 - a_\tau \cdot z^{|\tau|}) \cdot (1 - 2z)\, d(z, 1/F') \tag{3.3}$$

where $a_\tau = (F^n)'(x)$.

Remark. Whereas the coefficients in $d(z)$ are difficult to express directly, we can expand $1/\zeta(z)$, using Lemma 3(i), as a power series whose terms are far more accessible.

For the Feigenbaum tangent operator we have

$$\prod_{\tau} (1 - a_\tau z^{|\tau|}) = 1 - z(\alpha + \alpha^2) + \cdots$$

(because the fixed points 1, $1/\alpha$ have weights α^2, α, respectively).

By comparing the transfer operator with the weight function $1/F'$ with that with weight function $|1/F'|$ and recalling Lemmas 1 and 2(iii), we easily get that $d(z, 1/F')$ has *no zeros* in $|z| \leqslant 1$. In particular, we have the following:

Proposition 1. The zeros for $d(z)$ are the same as for

$$G(z) = \frac{d(z)}{d(z, 1/F')} = \prod_{\tau} (1 - a_\tau z^{|\tau|})(1 - 2z) \tag{3.4}$$

within the unit disc.

Remark. We observe that the expansion of $G(z)$ has a power series with real coefficients.

4. ANALYZING THE SPECTRUM

By Lemmas 4 and 2(iii), we know that the following two problems are equivalent:

(i) Locating eigenvalues β_i of L of modulus greater than unity.

(ii) Locating zeros $z_i = 1/\beta_i$ of $G(z)$ for $|z| \leqslant 1$.

We concentrate on the second version of the problem, and then the corresponding spectral results come easily from this correspondence.

4.1. Hyperbolicity and the Maximal Eigenvalue

To show that there is only one zero in $|z| \leqslant 1$, we want to compare the two complex functions $z \mapsto G(z)$ and $z \mapsto 1 + c_1 z$.

We can choose some $n \geqslant 1$ and write $G(z) = 1 + c_1 z + H(z) + K(z)$, where:

(i) $H(z) = c_2 z^2 + c_3 z^3 + \cdots + c_n z^n$ is a polynomial.

(ii) $K(z) = \sum_{r=n+1}^{+\infty} c_r z^n$ is a tail of the series.

By (1.1) we have the estimate

$$\sup_{|z|=1} |K(z)| \leqslant \sum_{r=n+1}^{+\infty} |c_r| \leqslant \sum_{r=n+1}^{+\infty} \frac{C^r \theta^{r(r-1-)/2}}{(1-\theta)\cdots(1-\theta^r)} = A_n \qquad (4.1)$$

and clearly $A_n = O(C^n \theta^{n^2})$, where the implied constant can be explicitly computed.

We recall the following fact from elementary complex analysis: If two functions f, g are analytic on a neighborhood of the unit disc and $|f(z) - g(z)| < |f(z)|$ wherever $|z| = 1$, then f and g have the same number of zeros in the disc.

We want to apply this elementary fact with the choices $f(z) = 1 + c_1 z$ and $g(z) = G(z)$ [and consequently $g(z) - f(z) = H(z) + K(z)$].

Proposition 2. If $|c_1| > 1$, then there exists $n \geqslant 1$, such that, providing, $\sup_{|z|=1} |K(z)| \leqslant (|c_1| - 1)/2$, say, then $G(z)$ has exactly one zero in $|z| \leqslant 1$.

Furthermore, we can choose

$$n = O\left(\left[\frac{\log(|c_1| - 1)}{\log\theta}\right]^{1/2}, \frac{\log(|c_1| - 1)}{C}\right)$$

where the implied constants can be explicitly computed.

Remark. (i) For the Feigenbaum tangent operator, $c_1 = \alpha^2 + \alpha - 2 > 1$ and the coefficients of polynomials of the form $H(z)$ are computed in ref. 1. For estimates relevant to θ, $C > 0$, see refs. 6 and 10.

(ii) Since the coefficients of G are real, the unique zero in $|z| \leqslant 1$ must lie on the real line (since complex zeros would appear as conjugate pairs).

4.2. Locating the Maximal Eigenvalue

We can now see that the approach of Artuso *et al.*[1] to computing δ is both rigorous and efficient.

Let $G_n(z) = 1 + c_1 z + \cdots + c_n z^n$ be the truncation of the series for $G(z)$ to n places. On a disc of radius $r > 0$ we have that $\|G - G_n\|_\infty = O(C^n \theta^{n^2})$.

Assume that $G(z)$ has only one zero z_0 in $|z| < r$ and no zeros on the circle $|z| = r$. For sufficiently large n the same will be true of G_n, and we denote the zero by z_n. Clearly,

$$z_0 = \frac{1}{2\pi i} \int_{|z|=r} z \cdot \frac{G'(z)}{G(z)}\, dz, \qquad z_n = \frac{1}{2\pi i} \int_{|z|=r} z \cdot \frac{G_n'(z)}{G_n(z)}\, dz$$

so that we can bound $|z_n - z_0|$ in terms of $\|G - G_n\|_\infty$.

We can therefore conclude:

Proposition 3. If $G(z)$ has a single zero z_0 in $|z| \leqslant r$, then this is approximated by zeros z_n for $G_n(z)$ with $|z_n - z_0| = O(C^n \theta^{n^2})$, where the implied constants can be explicitly computed.

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REFERENCES

1. R. Artuso, E. Aurell, and P. Cvitanovic, Recycling of strange sets II. Applications, preprint.
2. R. Bowen and O. Lanford, Zeta functions of restrictions of the shift transformation, *Proc. Symp. Pure Math. Am. Math. Soc.* **14**:43–50 (1970).
3. M. Campanino, H. Epstein, and D. Ruelle, On Feigenbaum's functional equation $g \circ g(\lambda x) + \lambda g(x) = 0$, *Topology* **21**:125–129 (1982).
4. J.-P. Eckmann and H. Epstein, Bounds on the unstable eigenvalue for period doubling, *Commun. Math. Phys.* **128**:427–435 (1990).
5. H. Epstein, New proofs of the existance of the Feigenbaum functions, *Commun. Math. Phys.* **106**:393–426 (1986).
6. H. Epstein, Fixed points of composition operators II, *Nonlinearity* **2**:305–310 (1988).
7. D. Fried, The zeta functions of Ruelle and Selberg I, *Ann. Sci. Ecole Norm. Sup.* **19**:491–517 (1986).
8. A. Grothendieck, Espaces nucléaires, *Mem. Am. Math. Soc.* **16** (1955).
9. K. Khanin, Y. Sinai, and E. Vul, Feigenbaum universality and the thermodynamic formalism, *Russ. Math. Surv.* **39**:1–40 (1984).
10. O. Lanford, Computer-assisted proofs in analysis, *Proc. Int. Cong. Math.* **1986**(II): 1385–1394.
11. D. Mayer, On a zeta-function related to the continued fraction transformation, *Bull. Soc. Math. France* **104**:195–203 (1976).
12. D. Ruelle, *Thermodynamic formalism* (Addison-Wesley, Reading, Massachusetts, 1978).
13. D. Ruelle, Zeta functions for expanding maps and Anosov flows, *Invent. Math.* **34**:231–242 (1976).
14. D. Sullivan, Quasi-conformal homeomorphisms in dynamics, topology and geometry, *Proc. Int. Cong. Math.* **1986**(II):1216–1228.